Manifestation of quantum chaos based on construction of an alternative method

Jun Ma, Xiao-shu Jiang, and Ren-kuan Yuan
Department of Physics, Nanjing University, Nanjing 210008, People's Republic of China
(Received 15 November 1996)

In this paper an alternative method has been performed to manifest chaos. We define a set of orthogonal states and construct some Hermitian operators in a kicked top model. We define phase angle basis states which are a particular linear superposition of the usual eigenstates of J_z. Each of these states is uniquely characterized by a different and discrete value which is called phase angle. Based on the basis states, some operators are given to show the probability distribution and the variance of the phase angle. Choosing SU(2) coherent states as initial states, we study the properties of the system. [S1063-651X(97)08004-5]

PACS number(s): 05.45.+b, 03.65.Sq

In recent years much work has been done to elucidate the quantum dynamics of classical chaotic systems and great progress has been made in studying chaos in a quantum system [1]. As proposed in Ref. [2], quantum chaos can be considered as a quantum statistical relaxation. According to such a point of view, we will investigate the temporal variation of a wave packet to study quantum chaos. There are various methods available for studies of the chaos in quantum mechanics. For example, uncertainty measure [3] and expectation value of a certain dynamical observable [4] are employed to manifest quantum chaos. The temporal variation of a wave packet has been studied with help of the information entropy [5]. But we will introduce an alternative method in this paper.

In this paper, we study a property of temporal variation of motion of a system. A nonintegrable Hamiltonian H is considered,

$$H=H^0+V, \tag{1}$$

where H^0 is an integrable Hamiltonian and V is perturbation. For a given initial state $|\psi(0)\rangle$, the time evolution is $|\psi(t)\rangle=U(t)|\psi(0)\rangle$; here, $U(t)$ is the unitary time evolution operator. The coherent representation (or say Husimi representation) and H^0 representation are always employed to analyze the time evolution state $|\psi(t)\rangle$. Although the coherent representation provides a powerful tool to study quantum chaos, the overcomplete coherent states chosen as a representation bring on some slight difficulties. If we expand the state $|\psi(t)\rangle$ in H^0 representation, we will obtain the expansion coefficients $c_i(t)e^{i\phi_i(t)}, i=1,2,\ldots N$, here we assume both $c_i(t)$ and $\phi_i(t)$ are real and the number of dimensions of the Hilbert space is N. A great deal of work has been done to analyze the distribution of $|c_i(t)|^2$ with the help of the information entropy [5]. But it is important to analyze the distribution of $\phi_i(t)$. If the above mentioned number of dimension of the Hilbert space N is infinite, the distribution of $|c_i(t)|^2$ may be confined in a certain energy region, then the distribution of $|c_i(t)|^2$ may be not random in all of Hilbert space. But one may find the random distribution of $\phi_i(t)$ with the help of a phase theory in such cases. Moreover, the random distribution of $c_i(t)$ does not indicate a random distribution of $\phi_i(t)$; we emphasize that distribution for both $c_i(t)$ and $\phi_i(t)$ should be analyzed in chaotic cases. So in this paper we introduce a method to study the distribution of $\phi_i(t)$.

We know that the phase theory was intensely utilized in quantum optics [6–9]. Pegg and Barnett introduced a set of orthogonal states of phase and an Hermitian phase operator that tends to the classical correspondence in classical limit [6–8]. The phase theory enables us to examine the phase properties of a system; moreover, the theoretical results can be tested in experiments [10,11]. So we think that it is necessary to introduce a general phase theory to manifest the quantum chaos. We corroborate this idea with the help of the kicked top model.

The kicked top model describes the dynamics of a large spin subject to a magnetic field and an impulsive interaction, and it is a suitable model for studying various problems [12,13]. Recently, attention has been devoted to observing the maintenance and loss of coherence of an initial coherent state in the kicked top [14]. The dynamical variable of the model is the angular momentum vector $\hbar\vec{J}=\hbar[\hat{J}_x,\hat{J}_y,\hat{J}_z]$, which obeys the commutation relation $[\hat{J}_i,\hat{J}_j]=i\varepsilon_{ijk}\hat{J}_k$. The Hamiltonian is

$$H(t)=(\hbar p/T)\hat{J}_x+(\hbar\lambda/2j)\hat{J}_z^2\sum_{-\infty}^{+\infty}\delta(t-nT). \tag{2}$$

We choose $p=\pi/2$ in all of this paper. This quantum system is generally analyzed in J_z representation $|j,m\rangle$. The normalized vector $\vec{V}=\vec{J}/j$ lies on the unit sphere in the classical limit $j\to\infty$. The time evolution of the vector $\vec{V}$ of the corresponding classical model can be obtained by the following map [12]:

$$\begin{aligned}X'&=X\cos(\lambda Y)+Z\sin(\lambda Y),\\ Y'&=X\sin(\lambda Y)-Z\cos(\lambda Y),\\ Z'&=Y.\end{aligned} \tag{3}$$

Because the vector $\vec{V}$ is unit vector in classical limit, the corresponding point (X,Y,Z) is on the unit sphere. The canonical coordinates of the unit sphere are $\cos(\zeta)$ and φ, $0\leqslant\zeta\leqslant\pi$; $0\leqslant\varphi\leqslant2\pi$, they are

$$\begin{aligned}I&=Z=\cos(\zeta),\\ \varphi&=\arctan(Y/X).\end{aligned} \tag{4}$$

To study quantum chaos we take the classical cases as guides. We project point (X,Y,Z) in the unit sphere onto a plane, i.e., (X,Y) plane. Then we can obtain the stroboscopic map in (X,Y) plane for various kicking strength. Figures 1(a)–1(c) display the stroboscopic maps for $\lambda=2.5$, 3.2, and 4, respectively. Here we only plotted the projection of the upper hemisphere $(Z>0)$. These plots show that the increasing kicking strength enlarges the chaotic region gradually.

Because a SU(2) coherent state is the quantum analog of a certain point in the classical unit sphere, it is necessary to choose a SU(2) coherent state as the initial state in quantum cases. The SU(2) coherent states $|\zeta,\varphi\rangle$ are [12]

$$|\zeta,\varphi\rangle=(1+\gamma\gamma^*)^{-j}e^{\gamma j_-}|j,j\rangle, \tag{5}$$

where $\gamma=e^{i\varphi}\tan(\zeta/2)$. Then

$$\langle j,m|\zeta,\varphi\rangle=(1+\gamma\gamma^*)^{-j}\gamma^{j-m}\left[\begin{pmatrix}2j\\ j-m\end{pmatrix}\right]^{1/2}. \tag{6}$$

Equation (6) shows that a coherent state is strongly localized and the continuous variable $\cos(\zeta)$ is the classical analog of the quantum number J_z/j when $j\to\infty$. So in a coherent state we also denote $\cos(\zeta)$ as the one in classical dynamics.

When the initial state $|\psi(0)\rangle$ is given in the kicked top model, its time evolution $|\psi(t)\rangle$ is generally discussed in J_z representation, $|j,-j\rangle,|j,-j+1\rangle,\ldots,|j,j\rangle$, where j is an integer or a half-integer. Now we introduce a particular space in the kicked top in parallel to the space used in the Pegg-Barnett theory. The particular space is based on introducing a finite $2j+1$ dimensional space Ψ spanned by the states $|j,-j\rangle$, $|j,-j+1\rangle,\ldots,|j,j\rangle$. Moreover, some Hermitian operators operating on this finite space are defined. The set of orthogonal phase states are defined by

$$|\theta_m\rangle=\frac{1}{\sqrt{2j+1}}\sum_{n=-j}^{j}\exp(in\theta_m)|j,n\rangle, \tag{7}$$

where the n is also an integer or a half-integer. The θ_m reads as

$$\theta_m=\theta_0+\frac{2m\pi}{2j+1},\quad m=0,1,2,\ldots,2j. \tag{8}$$

The value of θ_0 is arbitrary. It is known from Eqs. (7) and (8) that each of the basis states is linear superposition of the usual eigenstates of J_z and each of them is characterized by a different value of a discrete parameter θ_m. We call θ_m the phase angle.

When a SU(2) coherent state $|\zeta,\varphi\rangle$ is chosen as the initial state in the kicked top, its time evolution is $|\psi(nT)\rangle$,

$$|\psi(nT)\rangle=U^n(T)|\zeta,\varphi\rangle, \tag{9}$$

where the $U(T)$ is the Floquet operator $U(T)=\exp[-i(\lambda/2j)J_z^2]\exp[-ipJ_x/T]$. Then the probability distribution of phase angle $P(\theta_m)$ for $|\psi(nT)\rangle$ is defined as

$$P(\theta_m,nT)=\langle\theta_m|U^n(T)|\zeta,\varphi\rangle\langle\zeta,\varphi|U^{+n}(T)|\theta_m\rangle. \tag{10}$$

In the basis states, an Hermitian phase angle operator can be defined as

$$\hat{\phi}_\theta=\sum_{m=0}^{2j}\theta_m|\theta_m\rangle\langle\theta_m|,\quad \hat{\phi}_\theta|\theta_m\rangle=\theta_m|\theta_m\rangle. \tag{11}$$

The expectation value of the phase angle operator for the $|\psi(nT)\rangle$ is given by

$$\langle\hat{\phi}_\theta(nT)\rangle=\sum_{m=0}^{2j}\theta_m\langle\psi(nT)|\theta_m\rangle\langle\theta_m|\psi(nT)\rangle. \tag{12}$$

From the expectation values of $\hat{\phi}_\theta$ and of $\hat{\phi}_\theta^2$ one can obtain the variance of phase angle, which is written as F_1,

$$F_1=\langle(\Delta\hat{\phi}_\theta)^2\rangle=\langle\hat{\phi}_\theta^2\rangle-\langle\hat{\phi}_\theta\rangle^2. \tag{13}$$

In this paper, we make the particular choice of $\theta_0=-2j\pi/2j+1$. Then θ_m follows,

$$\theta_m=-\frac{2j\pi}{2j+1}+\frac{2\pi m}{2j+1},\quad m=0,1,\ldots,2j. \tag{14}$$

Choosing the SU(2) coherent state $|1,\pi/4\rangle$ as the initial state and using Eqs. (7)–(14) one can obtain the probability distribution of phase angle and the variance of the phase angle. From Fig. 1 we know that the classical point corresponding the coherent state $|1,\pi/4\rangle$ is trapped in a large regular region when $\lambda=2.5$, and that the point for $\lambda=3.2$ is embedded in a small stable inland. From Fig. 1(c), it is easy to see that the point for $\lambda=4$ is in a chaotic sea. Now we will see the results of $F_1(nT)$ for the initial state $|1,\pi/4\rangle$ in quantum mechanics. The results shown in Fig. 2, where curves 1, 2, and 3 are $F_1(nT)$ for $\lambda=2.5$, 3.2, and 4, respectively. It is shown from curve 1 for $\lambda=2.5$ that the variance of phase angle is small, which indicates that the structure of phase angle for the $|\psi(nT)\rangle$ is always localized. It is indicated from curve 2 for $\lambda=3.2$ that the variance of phase angle saturates at about 1 rad. It is easy to see that curve 3 rapidly reaches its saturation value near π, which indicates that the structure of the phase angle is delocalized in the long-time region.

If the same initial state is chosen, the results for probability distribution of the phase angle at special time are displayed in Fig. 3. Figure 3(a) shows the results for $\lambda=2.5$, where the dotted curve is the probability distribution of phase angle for the initial coherent state $|1,\pi/4\rangle$, and the solid curve is the results at $t=197T$. The two curves in Fig. 3(a) tell us that both the $|\psi(0)\rangle$ and $|\psi(197T)\rangle$ possess a regular structure of a phase angle. In Fig. 3(b), the dotted and solid curves show the results for $\lambda=3.2$ at $t=0$ and $120T$, respectively. From curve 2 in the Fig. 2, one can see that the relaxation process is over when $t=120T$, so Fig. 3(b) means that the kick with $\lambda=3.2$ cannot destroy the localized structure of the phase angle completely. The results for $\lambda=4$ are plotted in Fig. 3(c), where the dotted and solid are for $t=0$ and $30T$, respectively. Figure 3(c) shows that the initial regular structure of the phase angle is destroyed completely by the kicks. And the solid curve is Fig. 3(c) indicates that the probability distribution of the phase angle at $t=30T$ is already stochastic.

To analyze the phase probability distribution in detail we define the Shannon entropy as

$$S(nT)=-\sum_{m=0}^{2j}P(\theta_m,nT)\ln[P(\theta_m,nT)]. \tag{15}$$

The j is chosen as 70 in our calculation. We plot the numerical results of $S(nT)$ for $\lambda=4$ in Fig. 4, where one can see that the time evolution of S quickly reaches the saturation value which equals to 4.5 approximately and that the long-time $S(nT)$ stochastically fluctuates around the saturation value, which can be predicted by the random theory.

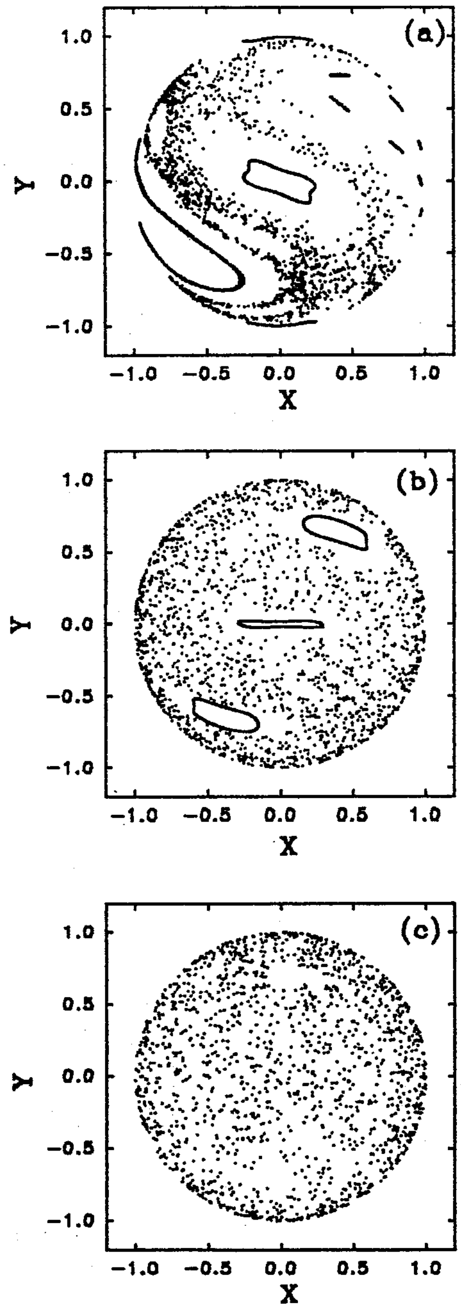

FIG. 1. Stroboscopic maps of phase space for the classical kicked top with $p=\pi/2$ and (a) $\lambda=2.5$; (b) $\lambda=3.2$; (c) $\lambda=4$. Above stroboscopic maps are for the hemisphere of $Z>0$.

If we assume that $P(\theta_m,nT)$ can be described by the χ_2^2 distribution

$$\chi_2^2(y)=\frac{1}{\langle y\rangle}\exp(-y/\langle y\rangle), \tag{16}$$

where the $\langle y\rangle=1/2j+1$. The ideal Shannon entropy can be obtained [13] as

$$S=-(2j+1)\int_0^\infty y\ \ln(y)\chi_2^2(y)dy=\ln(2j+1)-\Psi(2), \tag{17}$$

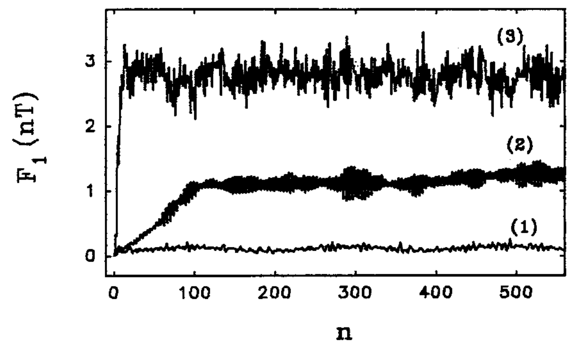

FIG. 2. Curves 1, 2, and 3 correspond to $F_1(nT)$ for $\lambda=2.5$, 3.2, and 4, respectively. Here $|\psi(0)\rangle=|1,\pi/4\rangle$ and $p=\pi/2$.

where Ψ is the digamma function. When $j=70$, the ideal S is 4.53. It can be concluded from Fig. 4 that the phase probability distribution for $\lambda=4$ is random in long-time regions.

In quantum optics the sine and cosine components of phase are intensely used, which can be measured in experiments [10,11]. So we also discussed both sine and cosine components of the phase angle in the kicked top model. From Eqs. (7), (8), and (11), through brief deduction, we can obtain two exponential operators $\exp(i\hat{\phi}_\theta)$ and $\exp(-i\hat{\phi}_\theta)^+$,

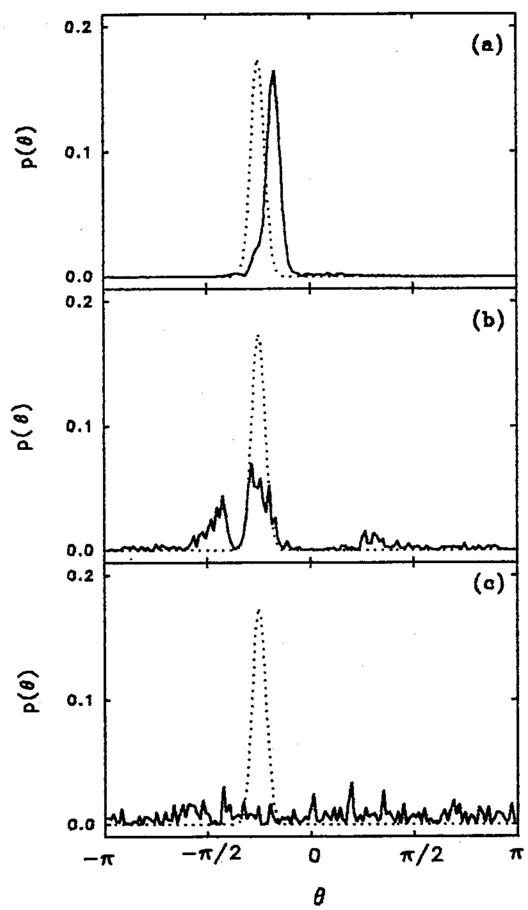

FIG. 3. When $|\psi(0)\rangle=|1,\pi/4\rangle$ and $p=\pi/2$, the results of the phase probability distribution is for (a) $\lambda=2.5$; (b) $\lambda=3.2$; (c) $\lambda=4$. In (a) the dotted line is for $n=0$ and the solid line is for $n=197$. In (b) the dotted line is for $n=0$ and the solid line is for $n=120$. In (c) the dotted line is for $n=0$ and the solid line is for $n=30$.

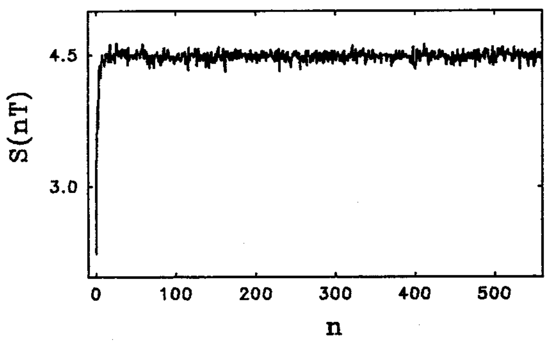

FIG. 4. $S(nT)$ for the initial state $|1,\pi/4\rangle$ with $\lambda=4$, $p=\pi/2$ and $j=70$.

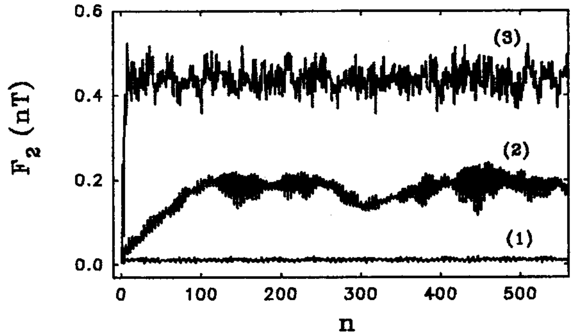

FIG. 5. Curves 1, 2, and 3 correspond to $F_2(nT)$ for $\lambda=2.5$, 3.2, and 4, respectively. Here $|\psi(0)\rangle=|1,\pi/4\rangle$ and $p=\pi/2$.

and the $\exp(i\hat{\phi}_\theta)$ are given by

$$\exp(i\hat{\phi}_\theta)=\exp[i(2j+1)\theta_0]|j,j\rangle\langle j-j| + \sum_{n=-j}^{j-1}|j,n\rangle\langle j,n+1|, \tag{18}$$

and it is obvious that

$$\exp(i\hat{\phi}_\theta)|j,n\rangle=\begin{cases}\exp[i(2j+1)\theta_0]|j,j\rangle, & n=-j\\ |j,n-1\rangle, & n\neq -j.\end{cases} \tag{19}$$

From the above-mentioned equation one can know that the exponential operators are unitary. So $\exp(-i\hat{\phi}_\theta)=\exp(i\hat{\phi}_\theta)^+$. According to the work by Pegg and Barnet [6–8], the $\cos(\hat{\phi}_\theta)$, $\cos^2(\hat{\phi}_\theta)$ and $\sin(\hat{\phi}_\theta)$, $\sin^2(\hat{\phi}_\theta)$ are given by

$$\cos(\hat{\phi}_\theta)=\tfrac{1}{2}\,[\exp(i\hat{\phi}_\theta)+\exp(-i\hat{\phi}_\theta)],$$
$$\sin(\hat{\phi}_\theta)=\frac{1}{2i}\,[\exp(i\hat{\phi}_\theta)-\exp(-i\hat{\phi}_\theta)] \tag{20}$$

and

$$\cos^2(\hat{\phi}_\theta)=\tfrac{1}{4}[\exp(2i\hat{\phi}_\theta)+\exp(-2i\hat{\phi}_\theta)+2],$$
$$\sin^2(\hat{\phi}_\theta)=-\tfrac{1}{4}[\exp(2i\hat{\phi}_\theta)+\exp(-2i\hat{\phi}_\theta)-2]. \tag{21}$$

It is easy to find the relation

$$\cos^2(\hat{\phi}_\theta)+\sin^2(\hat{\phi}_\theta)=1. \tag{22}$$

We define the variances of $\cos(\hat{\phi}_\theta)$ and of $\sin(\hat{\phi}_\theta)$ as follows:

$$F_2=\langle\cos^2(\hat{\phi}_\theta)\rangle-\langle\cos(\hat{\phi}_\theta)\rangle^2,$$
$$F_3=\langle\sin^2(\hat{\phi}_\theta)\rangle-\langle\sin(\hat{\phi}_\theta)\rangle^2. \tag{23}$$

From the same initial state $|1,\pi/4\rangle$, the time evolutions of F_2 and of F_3 for the kicked top model can be obtained. The numerical results for the variance of $\cos(\hat{\phi}_\theta)$, i.e., the time evolution of F_2 have been plotted in Fig. 5 where curves 1, 2, and 3 are the results for the $\lambda=2.5$, 3.2, and 4, respectively. Curve 1 shows that the variance of cosine component of phase angle for $\lambda=2.5$ is almost zero. In the case of $\lambda=3.2$, when $n=120$, $F_2(nT)$ will reach its saturation value, which is relatively large. Curve 3 for $\lambda=4$ tells us that the strong kick leads to great variances of the components of the phase angle quickly. One can see that Fig. 5 agrees with the results in Fig. 2 well.

At last we draw the main conclusion of this work. Through this work we know the initial coherent state possesses a regular structure of the phase angle, but perturbation will distort or destroy the regularity. In chaotic cases, due to the strong enough perturbation, the long-time probability distribution of the phase angle will be stochastic. Increasing the strength of perturbation enlarges the variances of $\hat{\phi}_\theta$, $\cos(\hat{\phi}_\theta)$, and $\sin(\hat{\phi}_\theta)$. Because the operators $\hat{\phi}_\theta$, $\cos(\hat{\phi}_\theta)$, and $\sin(\hat{\phi}_\theta)$ are realistic physical operators, their behavior may be observed in experiment. So our approach may provide a possibility of testing the quantum chaos experimentally. Generalizing the phase theory is interesting work which may open a new field about the study of quantum chaos.

[1] B. Eckhardt, Phys. Rep. **163**, 205 (1988).
[2] B. V. Chirikov, in *Chaos and Quantum Chaos*, edited by W. D. Heiss, Lecture Notes in Physics Vol. 411 (Springer-Verlag, New York, 1992).
[3] Y. Gu, Phys. Lett. A **149**, 95 (1990).
[4] A. Peres, Phys. Rev. A **30**, 1610 (1984).
[5] G. Casati, I. Guarneri, F. Izrailev, and R. Scharf, Phys. Rev. Lett. **64**, 5 (1990).
[6] D. T. Pegg and S. M. Barnett, Europhys. Lett. **6**, 483 (1988).
[7] D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989).
[8] S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989).
[9] R. Lynch, Phys. Rev. A **41**, 2841 (1990).
[10] J. W. Noh, A. Fougeres, and L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991).
[11] J. W. Noh, A. Fougeres, and L. Mandel, Phys. Rev. A **45**, 424 (1992).
[12] F. Haake, M. Kuś, and R. Scharf, Z. Phys. B **65**, 381 (1987).
[13] Karol Życzkowski, J. Phys. A **23**, 4427 (1990).
[14] X. Regez *et al.*, Comput. Phys. **10**, 38 (1996).